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# Two-soliton solutions of the Ernst equation

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**Abstract.** A connection between the Ernst equation and the chiral model on SL(2, $R$) is established. The Belinsky–Zakharov inverse method is used to solve the Ernst equation. Two-soliton solutions of the Ernst equation are given. It is shown that the two-soliton solution obtained from the Minkowski space–time is the NUT generalisation of the Kerr solution.

## 1. Introduction

Progress in general relativity is closely connected with investigation of exact solutions for Einstein's field equations. Because of the nonlinearity of Einstein's equations, it is extremely difficult to generate sufficiently general classses of solutions. The existence of a symmetry group simplifies the problem but does not remove the nonlinearity of the equations. In our paper we investigate stationary and axisymmetric gravitational fields. Recently, several generating techniques have been worked out for this case. Kinnersley and co-workers have presented in a series of papers (Kinnersley 1977, Kinnersley and Chitre 1978a, b, Hoenselaers *et al* 1979) an explicit representation for the infinite-dimensional Geroch group. Cosgrove (1980) has found a group Q outside the Geroch group which preserves asymptotical flatness of a space–time. Harrison (1978) and Neugebauer (1979) have applied a Bäcklund transformation to solve the field equations in the case of stationary and axisymmetric space–times. Finally, an 'inverse method' has been developed by Belinsky and Zakharov (1978, 1979) and Hauser and Ernst (1979a, b, 1980).

In our paper we apply the Belinsky–Zakharov method, which has been used by them for generating solutions of Einstein field equations in the stationary, axisymmetric case. However, we shall use the method to 'dress' a given Ernst potential i.e. to generate a new Ernst potential from a known one.

In § 2 we show a connection between the Ernst potential $\xi$ and the chiral model on SL(2, $R$). Since the method we use is very similar to the one described by Belinsky and Zakharov (1979), we present here only a short outline of it (§ 3). In § 4 we find and interpret two-soliton asymptotically flat solutions of the Ernst equation.

## 2. Axisymmetric and stationary gravitational fields as a chiral model

The Ernst equation is a compact complex formulation of the Einstein field equations in the case of axisymmetric and stationary metrics in the Papapetrou form. It can be

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derived from a Lagrangian density:

$$\mathscr{L} = \frac{\nabla\xi\nabla\xi^*}{(1-\xi\xi^*)^2}, \tag{2.1}$$

where $\nabla$ denotes the three-dimensional differential operator in cylindrical coordinates and $\xi$ is a function of $\rho$ and $z$. The Lagrangian (2.1) is SU(1, 1) symmetric. Let us rewrite it in an O(2, 1) symmetric form (SU(1, 1) ~ SO(2, 1)). This can be realised by a hyperbolic analogue (Woo 1977 and Mazur 1980) of the stereographic projection:

$$\xi = \frac{\mathrm{i}w^1 - w^2}{1+w^0}, \tag{2.2}$$

where the real vector $w^a$ $(a = 0, 1, 2)$ is normalised to 1:

$$\eta_{ab}w^a w^b = 1, \qquad \eta_{ab} = \mathrm{diag}(1, -1, -1). \tag{2.3}$$

The Lagrangian density (2.1) in the new parametrisation has the form:

$$\mathscr{L}_\sigma = \nabla w^a \nabla w^b \eta_{ab}. \tag{2.4}$$

Thus the problem can be treated as a nonlinear $\sigma$-model which can be related to a completely integrable chiral model on SL(2, $R$) with a constraint:

$$g = g^{\mathrm{T}}, \qquad g \in \mathrm{SL}(2, R). \tag{2.5}$$

Let $M := \{w\colon w^a w^b \eta_{ab} = 1\}$. Consider a map:

$$S: M \ni w \to g = \begin{pmatrix} w^0 + w^1 & w^2 \\ w^2 & w^0 - w^1 \end{pmatrix} \in \mathrm{SL}(2, R). \tag{2.6}$$

The map establishes a 1 : 1 connection between $M$ and a subset of symmetric matrices in SL(2, $R$). The choice of parametrisation for elements of the subset of SL(2, $R$) reduces the Lagrangian density of the chiral model on SL(2, $R$) to $\mathscr{L}_\sigma$ (2.4) and the constraint (2.3) is fulfilled by the condition $\det g = 1$.

## 3. Integrability of the chiral model

The model we consider leads to highly nonlinear field equations. In spite of this it is possible to find solutions. Namely we can take from the set of all solutions of the chiral field on GL(2, $R$) the solutions which belong to SL(2, $R$) and satisfy the condition (2.5). We can do it since the field equations for the chiral model on GL(2, $R$) are compatible with the conditions $\det g = 1$ and (2.5). Integrability of the model will be understood in this sense.

In the cylindrical coordinates $\rho$, $z$, $\varphi$ vectors $w^a$ and group elements $g$ depend on $\rho$ and $z$; therefore the field equations for the chiral model have the form:

$$(\rho g_{,\rho} g^{-1})_{,\rho} + (\rho g_{,z} g^{-1})_{,z} = 0. \tag{3.1}$$

Equation (3.1) was considered by Belinsky and Zakharov (1979). They elaborated a procedure which generates soliton solutions of (3.1) and preserves the required properties of $g$($\det g = 1$, reality and symmetry).

In our paper we apply their procedure. For the reader's convenience we give below a summary of the Belinsky–Zakharov method; details and proofs can be found in Belinsky and Zakharov (1979).

The generating technique for finding the soliton solutions of (3.1) consists in calculating a 'new' solution $g$ from a known 'old' solution, $g_0$. This can be done as follows. One defines potentials $U_0$, $V_0$:

$$U_0 = \rho g_{0,\rho} g_0^{-1}, \qquad V_0 = \rho g_{0,z} g_0^{-1}. \tag{3.2}$$

With the aid of $U_0$ and $V_0$ one can build a system of differential equations of first order (the Lax pair) for the $2\times 2$ (in general complex) matrix 'wavefunction' $\Psi_0$:

$$D_1\Psi_0 = \frac{\rho V_0 - \lambda U_0}{\lambda^2+\rho^2}\Psi_0, \qquad D_2\Psi_0 = \frac{\rho U_0 + \lambda V_0}{\lambda^2+\rho^2}\Psi_0. \tag{3.3}$$

Here $\lambda$ is a complex 'spectral parameter'. Differential operators $D_1$, $D_2$ are defined as follows:

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda, \qquad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda. \tag{3.4}$$

From the set of solutions of (3.3) one chooses such a 'wavefunction' $\Psi_0$ that

$$\Psi_0(\lambda = 0, \rho, z) = g_0(\rho, z). \tag{3.5}$$

Now $\Psi_0$ permits one to define two-dimensional vectors $\boldsymbol{m}^{(k)}$ and $\boldsymbol{N}^{(k)}$ ($k = 1, 2, \ldots, n$; $n \in N$):

$$\boldsymbol{m}^{(k)} = \Psi_0^{-1}(\lambda = \mu_k, \rho, z)\boldsymbol{m}_0^{(k)}, \tag{3.6}$$

$$\boldsymbol{N}^{(k)} = g_0 \boldsymbol{m}^{(k)}. \tag{3.7}$$

The vectors $\boldsymbol{m}_0^{(k)}$ form a system of arbitrary (in general complex) two-dimensional constant vectors.

The $\mu_k$ are functions defined by the formula:

$$\mu_k(\rho, z) = v_k - z + \varepsilon_k[(v_k - z)^2 + \rho^2]^{1/2}, \tag{3.8}$$

where parameters $v_k$ are in general complex and $\varepsilon_k = \pm 1$. With the aid of $\boldsymbol{m}^{(k)}$ and $\boldsymbol{N}^{(k)}$ one can construct a matrix:

$$(D^{-1})_{kl} = \boldsymbol{m}^{(k)T} g_0 \boldsymbol{m}^{(l)}(\rho^2 + \mu_k\mu_l)^{-1}, \qquad k, l = 1, 2, \ldots, n \tag{3.9}$$

which gives a new solution $\hat{g}$ of (3.1):

$$\hat{g} = g_0 - \sum_{k,l=1}^{n} D_{kl}\mu_k^{-1}\mu_l^{-1}\boldsymbol{N}^{(k)} \otimes \boldsymbol{N}^{(l)}. \tag{3.10}$$

If one assumes that $\boldsymbol{m}^{(k)}$ and $v_k$ are real, then the requirement for the reality of matrix $\hat{g}$ is automatically fulfilled. However, one can admit the case of complex $\boldsymbol{m}^{(k)}$ and $v_k$ as well. Then the reality of $\hat{g}$ is satisfied by the conditions:

$$\forall p \in \{1, 2, \ldots, n\} \quad \exists q \in \{1, 2, \ldots, n\}: v_p = v_q^*, \qquad \boldsymbol{m}_0^{(p)} = \boldsymbol{m}_0^{(q)*}, \qquad \varepsilon_p = \varepsilon_q. \tag{3.11}$$

A symmetry of $\hat{g}$ is obviously satisfied in (3.10). The determinant of the matrix $\hat{g}$ is different from 1. However, it turns out that when $n$ is an even number, the matrix:

$$g = -(\det \hat{g})^{1/2}\hat{g} \tag{3.12}$$

is a solution of (3.1) and $\det g = 1$.

Thus, by means of the Belinsky–Zakharov method one can find a new solution $g$ of the chiral model on SL(2, $R$). Now, the connection between the chiral model and the $\sigma$

model we established in § 2 permits us to map the new $g$ on $w^a(\rho, z)$ (2.6). The Ernst potential $\xi$ may then be obtained from (2.2).

## 4. Two-soliton solutions of the Ernst equation

We shall use the procedure described above to find the two-soliton solution ($n=2$) of the Ernst equation, starting from the Minkowski solution. For the Minkowski space–time $[(\xi-1)/(\varepsilon+1)=1]$ it is convenient to introduce the vectors $w^a (a=0,1,2)$ from the Ernst potential $\mathscr{E}$ which is connected with $\xi$ and $w^a$ by the formulae:

$$\mathscr{E}=\frac{\xi-1}{\xi+1}=-\frac{1}{w^0-w^2}+\mathrm{i}\frac{w^1}{w^0-w^2}. \tag{4.1}$$

From (2.6), (3.2) and (3.3) one can calculate $g_0$, $V_0$, $U_0$, $\Psi_0$:

$$g_0=\mathrm{diag}(-1,-1), \qquad V_0=U_0=0, \qquad \Psi_0\equiv g_0, \tag{4.2}$$

and by means of (3.6)–(3.10) and (3.12) one obtains a new solution $g$. A new potential $\xi$ is obtained from (2.6) and (2.2).

### *4.1. The case of real parameters*

In this case it is convenient to shift the origin of the coordinate $z$ to a point in which $v_1=v_2:=v$. If we introduce new parameters:

$$\begin{aligned} \boldsymbol{m}_0^{(1)\mathrm{T}}\boldsymbol{m}_0^{(1)}&=\sigma_1^2\\ \boldsymbol{m}_0^{(2)\mathrm{T}}\boldsymbol{m}_0^{(2)}&=\sigma_2^2\\ \boldsymbol{m}_0^{(1)\mathrm{T}}\boldsymbol{m}_0^{(2)}&=\sigma_1\sigma_2\cos(\varphi_1-\varphi_2) \end{aligned} \tag{4.3}$$

and prolate spheroidal coordinates:

$$\begin{aligned} \rho&=v(x^2-1)^{1/2}(1-y^2)^{1/2}\\ z&=vxy \end{aligned} \tag{4.4}$$

$\xi$ takes on the simplest form.

In the case $\varepsilon_1=\varepsilon_2=:\varepsilon$ we obtain the NUT generalisation of the Kerr solution:

$$\xi_{\text{K-NUT}}=\varepsilon\,\exp[\mathrm{i}(\varphi_1+\varphi_2)](px+\mathrm{i}qy), \tag{4.5}$$

where $p=\sin(\varphi_1-\varphi_2)$, $q=\cos(\varphi_1-\varphi_2)$.

The case of $\varepsilon_1=-\varepsilon_2:=\varepsilon$ can be obtained from (4.5) by complex cojugation.

### *4.2. The case of complex parameters*

Without loss of generality we can take $v_1=v_2^*:=\mathrm{i}v$. It is convenient to introduce real vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$:

$$\boldsymbol{m}_0^{(1)}=\boldsymbol{n}_1+\mathrm{i}\boldsymbol{n}_2 \tag{4.6}$$

and parameters $\sigma$, $b$, $\varphi_1$, $\varphi_2$:

$$\begin{aligned} \boldsymbol{n}_1^{\mathrm{T}}\boldsymbol{n}_1&=b^2\sigma^2\\ \boldsymbol{n}_2^{\mathrm{T}}\boldsymbol{n}_2&=\sigma^2\\ \boldsymbol{n}_1^{\mathrm{T}}\boldsymbol{n}_2&=b\sigma^2\cos(\varphi_1-\varphi_2). \end{aligned} \tag{4.7}$$

In 'oblate' spheroidal coordinates:

$$\rho = v(x^2+1)^{1/2}(1-y^2)^{1/2} \qquad z = vxy \tag{4.8}$$

a two-soliton solution of the Ernst equation is the NUT generalisation of the well known family:

$$\xi = (\hat{p}x + \mathrm{i}\hat{q}y)\,\mathrm{e}^{\mathrm{i}\beta} \tag{4.9}$$

where:

$$\hat{p} = -2bp/(a^4-4b^2p^2)^{1/2}, \qquad p = \sin(\varphi_1-\varphi_2), \qquad \beta = \sin^{-1}\frac{\tilde{p}qa^2-\tilde{q}p(b^2-1)}{(a^4-4b^2p^2)^{1/2}}, \qquad \hat{q}^2-\hat{p}^2 = 1. \tag{4.10}$$

Parameters $a$, $q$, $\tilde{p}$, $\tilde{q}$ are defined by the formulae:

$$a^2-b^2=1, \qquad p^2+q^2=1, \qquad \tilde{p} = \varepsilon\sin(\varphi_1+\varphi_2), \qquad \tilde{p}^2+\tilde{q}^2=1. \tag{4.11}$$

One obtains the solution (4.9) from the Kerr solution by the substitution $x \to \mathrm{i}x$, $p \to -\mathrm{i}\hat{p}$.

We have shown that the Ernst equation can be integrated by means of the Belinsky–Zakharov inverse method. The method permits one to construct the $n$-soliton solution (where $n$ is an even number) from a given known potential $\xi_0$.

The described connection between the Ernst equation and the chiral model on the SL(2, $R$) allows one to apply the non-soliton sector of the set of solutions of the Ernst equation.

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